Nyquist Theorem*

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Abstract

This module introduces the Nyquist theorem.

1 Introduction

Earlier you should have been exposed to the concepts behind sampling[1] and the sampling theorem. While learning about these ideas, you should have begun to notice that if we sample at too low of a rate, there is a chance that our original signal will not be uniquely defined by our sampled signal. If this happens, then there is no guarantee that we can correctly reconstruct[2] the signal. As a result of this, the **Nyquist Theorem** was created. Below, we will discuss just what exactly this theorem tells us.

2 Nyquist Theorem

We will let T equal our sampling period (distance between samples). Then let $\Omega_s = \frac{2\pi}{T}$ (sampling frequency in radians/sec). We have seen that if $f(t)$ is bandlimited to $[-\Omega_B, \Omega_B]$ and we sample with period $T < \frac{\pi}{\Omega_b} \Rightarrow \left(\frac{2\pi}{\Omega_s} < \frac{\pi}{\Omega_B} \Rightarrow \Omega_s > 2\Omega_B\right)$ then we can reconstruct $f(t)$ from its samples.

Theorem 1: Nyquist Theorem ("Fundamental Theorem of DSP")

If $f(t)$ is bandlimited to $[-\Omega_B, \Omega_B]$, we can reconstruct it **perfectly** from its samples

$$f_s[n] = f(nT)$$

for $\Omega_s = \frac{2\pi}{T} > 2\Omega_B$

$\Omega_N = 2\Omega_B$ is called the "**Nyquist frequency**" for $f(t)$. For perfect reconstruction to be possible

$$\Omega_s \geq 2\Omega_B$$

where Ω_s is the sampling frequency and Ω_B is the highest frequency in the signal.

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[1]"Sampling" <http://cnx.org/content/m10798/latest/>
[2]"Reconstruction" <http://cnx.org/content/m10788/latest/>

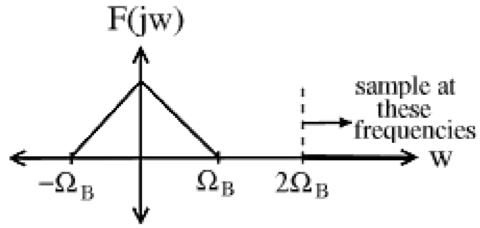

Figure 1: Illustration of Nyquist Frequency

Example 1: Examples:

- Human ear hears frequencies up to 20 kHz → CD sample rate is 44.1 kHz.
- Phone line passes frequencies up to 4 kHz → phone company samples at 8 kHz.

2.1 Reconstruction

The reconstruction formula in the time domain looks like

$$f(t) = \sum_{n=-\infty}^{\infty} \left(f_s[n] \frac{\sin\left(\frac{\pi}{T}(t-nT)\right)}{\frac{\pi}{T}(t-nT)} \right)$$

We can conclude, just as before, that

$$\forall n, n \in \mathbb{Z} : \left(\frac{\sin\left(\frac{\pi}{T}(t-nT)\right)}{\frac{\pi}{T}(t-nT)} \right)$$

is a basis[3] for the space of $[-\Omega_B, \Omega_B]$ bandlimited functions, $\Omega_B = \frac{\pi}{T}$. The expansion coefficient for this basis are calculated by sampling $f(t)$ at rate $\frac{2\pi}{T} = 2\Omega_B$.

> NOTE: The basis is also orthogonal. To make it orthonormal[4], we need a normalization factor of $\sqrt{T}$.

2.2 The Big Question

Exercise 1 ***(Solution on p. 3.)***

What if $\Omega_s < 2\Omega_B$? What happens when we sample below the Nyquist rate?

[Media Object][5]

[3]"Linear Algebra: The Basics" <http://cnx.org/content/m10734/latest/>
[4]"Orthonormal Basis Expansions" <http://cnx.org/content/m10760/latest/>
[5]This media object is a LabVIEW VI. Please view or download it at <http://cnx.org/content/m10791/2.6/NyquistPlot.llb>

Solutions to Exercises in this Module

Solution to Exercise 1 (p. 2)
Go through the steps: (see Figure 2)

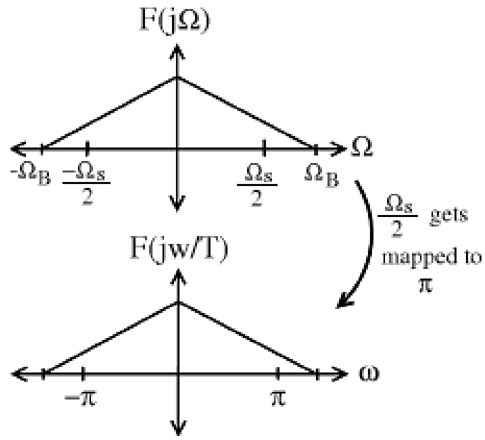

Figure 2

Finally, what will happen to $F_s\left(e^{i\omega}\right)$ now? To answer this final question, we will now need to look into the concept of aliasing[6].

[6]"Aliasing" <http://cnx.org/content/m10793/latest/>